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Orders of Strong Unicity Constants

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Given $f \in C(I)$, the growth of the strong unicity constant $M_n(f)$ for changing dimension is considered. Under appropriate hypotheses it is shown that $2n+1 \leqslant M_n(f) \leqslant \beta n^2$. Furthermore, relationships between certain Lebesgue constants and $M_n(f)$ are established.

1. INTRODUCTION

Let $C(\mathbf{X})$ denote the space of real valued, continuous functions on the compact set $\mathbf{X}$, and let $\mathbf{P}_{n+1} \subseteq C(\mathbf{X})$ be a Haar subspace of dimension $n+1$. Denote the uniform norm on $C(\mathbf{X})$ by $\|\cdot\|$.

For each $f \in C(\mathbf{X})$, with best approximation $B_n(f)$ from $\mathbf{P}_{n+1}$, there is a constant $r > 0$ such that for any $p \in \mathbf{P}_{n+1}$

$$\|p - B_n(f)\| \leqslant r(\|f - p\| - \|f - B_n(f)\|). \tag{1.1}$$

This inequality is the well-known strong unicity theorem [17].

DEFINITION 1. The strong unicity constant $M_n(f)$ is the smallest constant $r > 0$ such that (1.1) is valid for all $p \in \mathbf{P}_{n+1}$.

For example, if $\mathbf{P}_{n+1} = \Pi_n$, the space of polynomials of degree at most n, then $M_n(p) = 1$ if $p \in \Pi_n$ and $M_n(q) = 2n+1$ if q is a polynomial of degree exactly $n+1$ [6, 9].

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Several recent papers [1, 2, 4, 6, 9–11, 14, 18, 22] have been written on the subject of the dependence of $M_n(f)$ on f, n, and the domain $\mathbf{X}$. In particular, Ref. [4, 9, 10, 18, 22] examine the behavior of the sequence

$$\{M_n(f)\}_{n=0}^{\infty}. \tag{1.2}$$

Henry and Roulier [10] have conjectured that (1.2) is bounded if and only if f is a polynomial. References [4, 9, 22] all, to some extent, consider this conjecture.

The following definition is given in [9]:

DEFINITION 2. Let $f \in C(\mathbf{X})$, and suppose there exist positive constants α and β, a natural number N, and a positive real valued function c with domain the natural numbers satisfying

$$\alpha c(n) \leqslant M_n(f) \leqslant \beta c(n) \qquad \text{for all } n \geqslant N. \tag{1.3}$$

Then $M_n(f)$ is said to be of precise order $c(n)$.

Henry and Huff [9] established for $f(x) = 1/(x-a)$, $a \geqslant 2$, $x \in [-1, 1]$, that $M_n(f)$ is of precise order n. This is the first example of a non-polynomial function for which the precise order of $M_n(f)$ is known.

In the present paper the authors establish bounds on the order of growth of $M_n(f)$ for certain classes of functions.

Furthermore, relationships between $M_n(f)$ and the classical Lebesgue constant [19] are established.

2. Preliminaries

Throughout the remainder of this paper the domain of approximation $\mathbf{X}$ will be the interval $I = [-1, 1]$.

Let $f \in C(I) - \mathbf{P}_{n+1}$. Then it is known [1] that

$$M_n(f) = \left\{ \inf_{\substack{p \in \mathbf{P}_{n+1} \\ \|p\| = 1}} \max_{x \in E_{n+1}(f)} \operatorname{sgn} e_n(f)(x)\, p(x) \right\}^{-1}, \tag{2.1}$$

where $e_n(f) = f - B_n(f)$ and

$$E_{n+1}(f) = \{x \in I : |e_n(f)(x)| = \|e_n(f)\|\}. \tag{2.2}$$

Hereafter $\mathbf{P}_{n+1} = \Pi_n$. The following three theorems are utilized in the subsequent analysis.

THEOREM 1 (Cline [6]). *Let* $f \in C(I) - \Pi_n$, *and let* $\{x_k\}_{k=0}^{n+1}$ *be a Chebyshev alternation for* $e_n(f)$. *Define* $q_{in} \in \Pi_n$ *by* $q_{in}(x_k) = \operatorname{sgn} e_n(f)(x_k)$ *for* $k = 0, 1, \ldots, n+1$, $k \neq i$, *and* $i = 0, 1, \ldots, n+1$. *Then* $M_n(f) \leqslant \max_{0 \leqslant i \leqslant n+1}\{\|q_{in}\|\}$.

This theorem is extended in [10] where it is noted that if $e_n(f)$ has exactly $n+2$ extreme points, then

$$M_n(f) = \max_{0 \leqslant i \leqslant n+1}\{\|q_{in}\|\}. \tag{2.3}$$

THEOREM 2 (Rowland [21]). *Let* $f \in C[-1, 1]$, $f'' \in C(-1, 1)$, *and* $f^{(n+1)}(x) \neq 0$ *for* $x \in [-1, 1]$. *Also let* $-1 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$ *be the ordering of* $E_{n+1}(f)$. *Then*

(a) *if* $f^{(n+1)}$ *is positive and strictly increasing on* I, *then*

$$z_k < x_k < \xi_k, \qquad k = 1, 2, \ldots, n,$$

and

(b) *if* $f^{(n+1)}$ *is positive and strictly decreasing on* I, *then*

$$\xi_{k-1} < x_k < z_k, \qquad k = 1, 2, \ldots, n,$$

where

$$z_k = \cos\left(\frac{n+1-k}{n+1}\right)\pi \qquad \text{and} \qquad \xi_k = \cos\left(\frac{n-k}{n}\right)\pi. \tag{2.4}$$

Theorem 2 is a special case of Theorem 3.3 in [21] and ensures for a certain class of functions that the extreme points of $e_n(f)$ are separated by the extreme points of the Chebyshev polynomials T_n and T_{n+1} of degrees n and $n+1$, respectively.

THEOREM 3 (Bartelt and Schmidt [4]). *If* $f \in C(I) - \Pi_n$, *then*

$$M_n(f) = \max_{p \in \Pi_n}\{\|p\| : \operatorname{sgn} e_n(f)(x)\, p(x) \leqslant 1 \text{ for } x \in E_{n+1}(f)\}. \tag{2.5}$$

3. THE ORDER OF GROWTH OF $M_n(f)$

For any alternant $\{x_0, \ldots, x_{n+1}\} \subseteq E_{n+1}(f)$, define $Q_{n+1} \in \Pi_{n+1}$ by

$$Q_{n+1}(x_i) = \operatorname{sgn} e_n(f)(x_i), \qquad i = 0, \ldots, n+1. \tag{3.1}$$

LEMMA 1. *Let* $\{x_0, x_1, \ldots, x_{n+1}\}$ *be an alternant for* $e_n(f)$. *Define* $\{q_{in}\}_{i=0}^{n+1}$ *as in Theorem* 1. *Then*

$$q_{in}(x) = Q_{n+1}(x) - a_{n+1} \prod_{\substack{j=0 \\ j \neq i}}^{n+1} (x - x_j), \tag{3.2}$$

$i = 0, 1, \ldots, n+1$, *where* a_{n+1} *is the coefficient of* x^{n+1} *in* Q_{n+1}.

The proof of Lemma 1 follows immediately from the definition of q_{in} in Theorem 1 and (3.1). ∎

THEOREM 4. *For* $n \geqslant 1$ *let* $f^{(n+2)} \in C(I)$ *and suppose* $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \neq 0$ *on* I. *Then*

$$\max\{\|q_{0n}\|, \|q_{n+1,n}\|\} > 2n + 1$$

and thus

$$M_n(f) > 2n + 1.$$

Proof. First assume $f^{(n+1)}(x) \cdot f^{(n+2)}(x) > 0$ on I. By replacing f with $(-f)$ if necessary we may assume that $f^{(n+1)}(x) > 0$ on I. Let $g(x) = a_{n+1}x^{n+1}$, where a_{n+1} is defined in Lemma 1. Clearly

$$\|e_n(g)\| = \|g - B_n(g)\| = \frac{|a_{n+1}|}{2^n} \|T_{n+1}\| = \frac{|a_{n+1}|}{2^n}.$$

But then (3.1) and the theorem of de LaVallée Poussin [5, p. 77] imply that

$$\|e_n(g)\| \geqslant \min_{0 \leqslant i \leqslant n+1} |Q_{n+1}(x_i)| = 1.$$

This inequality now implies that

$$|a_{n+1}| \geqslant 2^n. \tag{3.3}$$

Theorem 2 (part a), Lemma 1, and (3.3) imply that

$$\begin{aligned} |q_{0n}(-1)| &= \left| Q_{n+1}(-1) - a_{n+1} \prod_{j=1}^{n+1} (-1 - x_j) \right| \\ &\geqslant 2^n \prod_{j=1}^{n+1} (1 + x_j) - 1 \\ &> 2^{n+1} \prod_{j=1}^{n} (1 + z_j) - 1 \\ &= \frac{2}{n+1} |T'_{n+1}(-1)| - 1 \\ &= 2n + 1. \end{aligned}$$

Similarly if $f^{(n+1)}(x) \cdot f^{(n+2)}(x) < 0$ on I then an application of Theorem 2 (part b) yields

$$|q_{n+1,n}(1)| > 2n + 1.$$

The conclusion of the theorem follows from (2.3). ∎

In light of the analysis given in [9] for $f(x) = 1/(x-a)$, $a \geqslant 2$, $x \in I$, the order of the lower bound given in Theorem 4 is sharp.

LEMMA 2. *Let $f \in C^\infty(I)$ and suppose that there exists a constant α such that for all n sufficiently large*

$$\left| \frac{f^{(n+1)}(\zeta)}{f^{(n+1)}(\eta)} \right| \leqslant \alpha \qquad \textit{for all} \quad \zeta, \eta \in I. \tag{3.4}$$

Then

(a) *$\|q_{0n}\|$ and $\|q_{n+1,n}\|$ are both of order n;*

and

(b) *$\max\{\|q_{0n}\|, \|q_{n+1,n}\|\}$ is of precise order n, where q_{0n} and $q_{n+1,n}$ are defined as in Theorem* 1.

Proof. Assume $f^{(n+1)}(x) \cdot f^{(n+2)}(x) > 0$ on I and (3.4). Then by the definition of q_{0n}

$$q_{0n}(x) = \frac{e_n(f)(x)}{\|e_n(f)\|} - \frac{f^{(n+1)}(\eta)}{(n+1)!\,\|e_n(f)\|} \prod_{i=1}^{n+1} (x - x_i) \tag{3.5}$$

for some $\eta \in (-1, 1)$. Thus by the proof of Theorem 4 $\|q_{0n}\|$ is of precise order n if and only if

$$\left| \frac{f^{(n+1)}(\eta)}{(n+1)!\,\|e_n(f)\|} \prod_{i=1}^{n+1} (x - x_i) \right| = O(n). \tag{3.6}$$

By replacing f by $(-f)$ if necessary we may assume that $f^{(n+1)}(x) > 0$ on I. Since for some $\zeta \in I$, $\|e_n(f)\| = |f^{(n+1)}(\zeta)/2^n(n+1)!|$ [16, p. 78], hypothesis (3.4) implies the left side of (3.6) is bounded by

$$(1/\alpha)\, 2^n \prod_{i=1}^{n+1} |x - x_i|. \tag{3.7}$$

If $x \in [-1, x_1]$, then (2.4) implies that

$$(1/\alpha)\, 2^n \prod_{i=1}^{n+1} |x - x_i| \leqslant (1/\alpha)\, 2^n \prod_{i=1}^{n} |x - \xi_i|\, |x - 1|$$
$$\leqslant (2/\alpha) \frac{|T_n'(x)|}{n} (x - 1)^2$$
$$\leqslant K_1 n.$$

If $x \in [x_n, 1]$, then again using (2.4),

$$(1/\alpha)\, 2^n \prod_{i=1}^{n+1} |x - x_i| \leqslant (1/\alpha)\, 2^n \prod_{i=1}^{n} |x - z_i|\, |x - 1|$$
$$\leqslant (1/\alpha) \frac{|T_{n+1}'(x)|}{n+1} |x^2 - 1|$$
$$\leqslant (2/\alpha).$$

Finally if $x \in (x_j, x_{j+1})$ for $1 \leqslant j \leqslant n-1$, then

$$(1/\alpha)\, 2^n \prod_{i=1}^{n+1} |x - x_i| \leqslant (1/\alpha)\, 2^n \prod_{i=1}^{j} |x - \xi_{i-1}| \prod_{i=j+1}^{n} |x - \xi_i|\, |x - 1|$$
$$= (2/\alpha) \frac{|T_n'(x)|}{|x - \xi_j|} |x^2 - 1|$$
$$= (2/\alpha)(1/n)\, |(1 - \tau^2)\, T_n''(\tau) - 2\tau T_n'(\tau)|$$
$$= (2/\alpha)(1/n)\, |\tau T_n'(\tau) + n^2 T_n(\tau)|$$
$$\leqslant K_2 n.$$

Therefore for all $x \in I$

$$(1/\alpha)\, 2^n \prod_{i=1}^{n+1} |x - x_i| \leqslant Kn$$

for some positive constant K independent of n. Consequently $\|q_{0n}\|$ is of precise order n. Next as in (3.5)

$$q_{n+1,n}(x) = \frac{e_n(f)(x)}{\|e_n(f)\|} - \frac{f^{(n+1)}(\bar{\eta})}{(n+1)!\, \|e_n(f)\|} \prod_{i=0}^{n} (x - x_i),$$

for some $\bar{\eta} \in (-1, 1)$. A minor modification of the argument below (3.6) shows that

$$\left| \frac{f^{(n+1)}(\bar{\eta})}{(n+1)!\, \|e_n(f)\|} \prod_{i=0}^{n} (x - x_i) \right| \tag{3.8}$$

is of order n and consequently $\|q_{n+1,n}\|$ is $O(n)$.

Since no Chebyshev extreme points separate x_n and x_{n+1} under the assumption that $f^{(n+1)}(x) \cdot f^{(n+2)}(x) > 0$ on I, the arguments utilized to establish that the precise order of $\|q_{0n}\|$ is n do not extend to $\|q_{n+1,n}\|$.

Similarly if $f^{(n+1)}(x) \cdot f^{(n+2)}(x) < 0$ on I, then the theorem follows from interchanging the roles of q_{0n} and $q_{n+1,n}$ and then following a parallel argument to that used above. ▌

The next theorem provides asymptotic estimates to $M_n(f)$ for a class of functions in $C(I)$.

THEOREM 5. *Under the hypothesis of Lemma* 2 *there exist positive constants* K_1 *and* K_2 *such that*

$$K_1 n \leqslant M_n(f) \leqslant K_2 n^2. \tag{3.9}$$

Proof. First note for $0 \leqslant i, j \leqslant n+1$ that

$$(x_i - x_j)\, Q_{n+1}(x) = (x - x_j)\, q_{jn}(x) - (x - x_i)\, q_{in}(x), \qquad x \in I, \tag{3.10}$$

follows directly from (3.2). Furthermore (3.4) implies that $f^{(n+1)}(x) \neq 0$ on I and thus $-1 = x_0 < x_1 < \cdots < x_{n+1} = 1$ are the extreme points for $e_n(f)$. Therefore, letting $i = 0$ and $j = n+1$ in (3.10) we obtain

$$2Q_{n+1}(x) = (x+1)\, q_{0n}(x) - (x-1)\, q_{n+1,n}(x), \qquad x \in I. \tag{3.11}$$

Equation (3.11) and Lemma 2 imply that $\|Q_{n+1}\|$ is $O(n)$. Equation (3.10) implies for $x \neq x_i$ that

$$q_{in}(x) = \frac{(x^2 - 1)}{2} \frac{q_{n+1,n}(x) - q_{0n}(x)}{(x - x_i)} + Q_{n+1}(x),$$

for $i = 1, \ldots, n$. Therefore, since $q_{n+1,n}(x_i) = q_{0n}(x_i) = \operatorname{sgn} e_n(f)(x_i)$, $i = 1, 2, \ldots, n$, the mean value theorem implies that

$$\begin{aligned} q_{in}(x) = {} & \zeta_x [q_{n+1,n}(\zeta_x) - q_{0n}(\zeta_x)] \\ & + (\zeta_x^2 - 1)[q'_{n+1,n}(\zeta_x) - q'_{0n}(\zeta_x)](\tfrac{1}{2}) \\ & + Q_{n+1}(x), \qquad \text{for} \quad i = 1, 2, \ldots, n. \end{aligned} \tag{3.12}$$

Since $q_{n+1,n}$ and q_{0n} are both $O(n)$, the middle term on the right side of (3.12) is $O(n^2)$ [15, p. 39]. Therefore (3.12) and (3.11) imply that

$$\|q_{in}\| = O(n^2), \qquad \text{for} \quad i = 1, 2,..., n. \tag{3.13}$$

Thus Lemma 2 and (2.3) imply (3.9). ∎

EXAMPLE 1. Let $f_1(x) = e^{\alpha x}$ for any real α, and let $f_2(x) = \cos x/2 + \sin x/2$. Then Theorem 5 applies to these functions and (3.9) is valid for both f_1 and f_2.

Hypothesis (3.4) restricts the class of functions to which Theorem 5 is applicable; however, such constraints which locate the extreme points of the error function $e_n(f) = f - B_n(f)$ are essential in estimating $M_n(f)$.

In Section 3 of [10] a function f is constructed for which

$$M_{n_i}(f) \geqslant 2^{n_i/2}, \qquad i = 1, 2,...,$$

where $\{n_i\}_{i=1}^{\infty}$ is an increasing sequence of nonnegative integers. Theorem 5 shows that for the defined class of functions the growth of $M_n(f)$ cannot be so dramatic.

4. LEBESGUE AND STRONG UNICITY CONSTANTS

In the present section we establish relationships between certain strong unicity constants and corresponding Lebesgue constants.

DEFINITION 3. Given any $n + 1$ distinct points $\{x_i\}_{i=0}^{n}$ the corresponding *Lebesgue function* $\lambda_n(x)$ is defined by

$$\lambda_n(x) = \sum_{i=0}^{n} |L_i(x)|,$$

where

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \qquad i = 0,..., n$$

are the standard Lagrange polynomials determined by $\{x_i\}_{i=0}^{n}$. The *Lebesgue constant* λ_n is defined by

$$\lambda_n = \|\lambda_n(\cdot)\| = \max_{-1 \leqslant x \leqslant 1} |\lambda_n(x)|.$$

For a discussion of Lebesgue constants see [19, pp. 87–101]. Theorem 6 (below) shows that the strong unicity constant, for any function whose error curve $e_n(f)$ has exactly $n+2$ extreme points, equals the largest of the Lebesgue constants determined by the point sets obtained by omitting one point at a time from $E_{n+1}(f)$.

First we state a lemma bounding $M_n(f)$. A proof of Lemma 3 below is essentially contained in [8], and the lemma is a special case of a lemma stated in [13, Lemma 1]. For completeness we do include a short, explicit proof of Lemma 3.

LEMMA 3. *Let $f \in C(I) - \Pi_n$ and let $\mathscr{A}_{n+1} = \{x_0, \ldots, x_{n+1}\}$ be an alternant for $e_n(f)$. Furthermore let λ^j_{n+1} denote the Lebesgue constant for the set $\mathscr{A}_{n+1} - \{x_j\}$ for $j = 0, 1, \ldots, n+1$. Then*

$$M_n(f) \leqslant \max_{0 \leqslant j \leqslant n+1} \lambda^j_{n+1}. \tag{4.1}$$

Proof. First note that $q_{jn}(x) = \sum_{i=0,\, i \neq j}^{n+1} \operatorname{sgn} e_n(f)(x_i)\, L^j_i(x)$ for $j = 0, 1, \ldots, n+1$, where q_{jn} is defined as in Theorem 1, and where

$$L^j_i(x) = \prod_{\substack{l=0 \\ l \neq i,j}}^{n+1} \frac{x - x_l}{x_i - x_l}, \qquad i = 0, 1, \ldots, n+1,\ i \neq j,$$

are the Lagrange polynomials determined by the set $\mathscr{A}_{n+1} - \{x_j\}$. Thus $\|q_{jn}\| \leqslant \lambda^j_{n+1}$, for $j = 0, 1, \ldots, n+1$. Therefore by Theorem 1

$$M_n(f) \leqslant \max_{0 \leqslant j \leqslant n+1} \lambda^j_{n+1}. \quad \blacksquare$$

THEOREM 6. *For $f \in C(I)$, suppose that $E_{n+1}(f)$ contains exactly $n+2$ points $\{x_i\}_{i=0}^{n+1}$. Let λ^j_{n+1} denote the Lebesgue constant determined by $E^j_{n+1} = E_{n+1}(f) - \{x_j\}$, $j = 0, \ldots, n+1$. Then*

$$M_n(f) = \max_{0 \leqslant j \leqslant n+1} \lambda^j_{n+1}. \tag{4.2}$$

Proof. Let $\bar{x}_j$ be a point in I at which $\lambda^j_{n+1} = |\lambda^j_{n+1}(\bar{x}_j)|$, and define p^j_n by

$$p^j_n(x) = \sum_{\substack{i=0 \\ i \neq j}}^{n+1} \operatorname{sgn} L^j_i(\bar{x}_j)\, L^j_i(x) \qquad \text{for} \quad x \in I. \tag{4.3}$$

As usual $e_n(f) = f - B_n(f)$. If $\operatorname{sgn} p^j_n(x_j)\, e_n(f)(x_j) > 0$, define $\bar{p}^j_n(x) = -p^j_n(x)$ for $x \in I$; otherwise define $\bar{p}^j_n(x) = p^j_n(x)$ for $x \in I$. Then $\|\bar{p}^j_n\| = \lambda^j_{n+1}$ for $j = 0, 1, \ldots, n+1$. Also from (4.3) $\bar{p}^j_n(x_k) = \pm \operatorname{sgn} L^j_k(\bar{x}_j)$ for

$k = 0, 1, \ldots, n+1$, $k \neq j$. Furthermore, the construction above ensures that $\bar{p}_n^j(x_j) \operatorname{sgn} e_n(f)(x_j) < 0$. Therefore

$$\bar{p}_n^j \in \{p \in \Pi_n : \operatorname{sgn} e_n(f)(x)\, p(x) \leqslant 1, x \in E_{n+1}(f)\}.$$

Hence Theorem 3 implies that

$$\|\bar{p}_n^j\| \leqslant M_n(f) \qquad \text{for} \quad j = 0, 1, \ldots, n+1.$$

Consequently

$$\max_{0 \leqslant j \leqslant n+1} \lambda_{n+1}^j \leqslant M_n(f). \tag{4.4}$$

An application of Lemma 3 completes the proof of this theorem. ∎

The following theorem relates the strong unicity constant $M_n(f)$ to λ_{n+1}, the Lebesgue constant determined by all of

$$E_{n+1}(f) = \{x_0, x_1, \ldots, x_{n+1}\}.$$

THEOREM 7. *Let $f \in C(I)$, let $E_{n+1}(f)$ contain exactly the $n+2$ points $x_0 < x_1 < \cdots < x_{n+1}$ and let λ_{n+1} be the corresponding Lebesgue constant. Then*

$$\text{(i)} \quad \|Q_{n+1}\| \leqslant \frac{4}{x_{n+1} - x_0} M_n(f) \tag{4.5}$$

and

$$\text{(ii)} \quad \lambda_{n+1} \leqslant \left(\frac{4}{x_{n+1} - x_0} + 1\right) M_n(f), \tag{4.6}$$

where Q_{n+1} is defined as in (3.1).

Proof. First let a_{n+1} denote the coefficient of x^{n+1} in Q_{n+1}. Since

$$Q_{n+1}(x) = \sum_{i=0}^{n+1} \operatorname{sgn} e_n(f)(x_i) \prod_{\substack{j=0 \\ j \neq i}}^{n+1} \left(\frac{x - x_j}{x_i - x_j}\right)$$

then

$$a_{n+1} = \sum_{i=0}^{n+1} \frac{\operatorname{sgn} e_n(f)(x_i)}{\prod_{j=0, j \neq i}^{n+1} (x_i - x_j)}.$$

Furthermore since $\operatorname{sgn} \prod_{j=0, j \neq i}^{n+1} (x_i - x_j) = (-1)^{n-i+1}$ then

$$|a_{n+1}| = \sum_{i=0}^{n+1} \frac{1}{\prod_{j=0, j \neq i}^{n+1} |x_i - x_j|}.$$

Next by (3.10)

$$Q_{n+1}(x) = \frac{(x - x_0)\, q_{0n}(x) - (x - x_{n+1})\, q_{n+1,n}(x)}{x_{n+1} - x_0}. \tag{4.7}$$

Therefore

$$\begin{aligned} \|Q_{n+1}\| &\leqslant \frac{4}{x_{n+1} - x_0} \max\{\|q_{0n}\|, \|q_{n+1,n}\|\} \\ &\leqslant \frac{4}{x_{n+1} - x_0} M_n(f). \end{aligned}$$

Also

$$\begin{aligned} \lambda_{n+1}(x) &= \sum_{i=0}^{n+1} \prod_{\substack{j=0 \\ j \neq i}}^{n+1} \left| \frac{x - x_j}{x_i - x_j} \right| \\ &\leqslant \max_i \prod_{\substack{j=0 \\ j \neq i}}^{n+1} |x - x_j| \sum_{i=0}^{n+1} \frac{1}{\prod_{j=0, j \neq i}^{n+1} |x_i - x_j|} \\ &= \max_i \prod_{\substack{j=0 \\ j \neq i}}^{n+1} |x - x_j|\, |a_{n+1}| \\ &= \max_i |Q_{n+1}(x) - q_{in}(x)| \\ &\leqslant \|Q_{n+1}\| + \max_i \|q_{in}\| \\ &\leqslant \left(\frac{4}{x_{n+1} - x_0} + 1 \right) M_n(f). \quad \blacksquare \end{aligned}$$

Remark. If $x_{n+1} - x_0 \geqslant \delta > 0$ for all n then (4.5) and (4.6) of Theorem 7 can be replaced by

$$\text{(i)} \quad \|Q_{n+1}\| \leqslant \frac{4}{\delta} M_n(f)$$

and

$$\text{(ii)} \quad \lambda_{n+1} \leqslant \left(\frac{4}{\delta} + 1 \right) M_n(f).$$

Furthermore if $x_0 = -1$ and $x_{n+1} = 1$ then by (4.7), $Q_{n+1}(x)$ is a convex combination of $q_{0n}(x)$ and $q_{n+1,n}(x)$, and thus (4.5) and (4.6) can be further improved yielding the following:

COROLLARY. *Let $f \in C(I)$ and let E_{n+1} contain exactly the $n+2$ points $-1 = x_0 < x_1 < \cdots < x_{n+1} = 1$. Then*

$$\text{(i)} \quad \|Q_{n+1}\| \leqslant M_n(f)$$

and

$$\text{(ii)} \quad \lambda_{n+1} \leqslant 2M_n(f).$$

A comparison of Theorem 6 and the remark after Theorem 7 reveals the following observation: for functions satisfying the required hypotheses, the maximum of the Lebesgue constants obtained by removing one point at a time from the extremal set $E_{n+1}(f)$, grows at least as fast as the Lebesgue constant determined by all of $E_{n+1}(f)$ as n tends to infinity.

The following example show that the orders of growth of $M_n(f) = \max_{0 \leqslant j \leqslant n+1} \lambda_{n+1}^j$ and λ_{n+1} may differ significantly.

EXAMPLE 2. Let f be a polynomial of degree exactly $N+1$. If approximation is from Π_N, then as previously noted, $M_N(f) = 2N+1$. The extremal set $E_{N+1}(f)$ for the error function $e_N(f)$ consists of precisely the $N+2$ extreme points of T_{N+1}. Therefore the precise order of $\lambda_{N+1}$is $\log(N+1)$ [7]. Thus the orders of $M_n(f) = \max_{0 \leqslant j \leqslant n+1} \lambda_{n+1}^j$ and λ_{n+1} may differ significantly.

5. OBSERVATIONS AND CONCLUSIONS

In the preceding sections the growth of $M_n(f)$ for certain $f \in C(I)$ is examined. Explicit relationships between strong unicity constants and Lebesgue constants are established. Furthermore, bounds on the rate of growth of $M_n(f)$ are developed.

It would be desirable to establish the precise order of $M_n(f)$ for functions satisfying the hypothesis of Theorem 5. It would also be of interest to find classes of functions for which $M_n(f)$ and λ_{n+1} are of the same precise order.

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